State Selection in Accelerated Systems

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(February 1, 2008)

The problem of state selection when multiple metastable states compete for occupation is considered for systems that are accelerated far from equilibrium. The dynamics of the supercurrent in a narrow superconducting ring under the influence of an external electric field is used to illustrate the general phenomenology.

PACS: 02.50.Ey,05.20.-y,05.40.+j

Many systems when driven far from equilibrium encounter instabilities that lead towards new states or phases. Frequently there exist multiple states that can be selected following the onset of the instability. The determination of the particular state that is selected is a complex problem of fundamental interest in a wide variety of fields [1,2]. In addition to the complexity associated with the presence of multiple competing states, if the system is accelerated so that the environment evolves in time, then the state that is selected can depend in an important way on the driving force. The focus of this paper is on state-selection in "accelerated" systems when multiple metastable states compete for occupation.

For the purposes of this paper an accelerated system is defined as one for which a control parameter is varied in time so that the system gradually progresses from stable to unstable regimes. For example, in a narrow superconducting ring [3–6] under the influence of a constant electromotive force, the superconducting electrons are accelerated by the electric field and the supercurrent increases with time until the critical current is reached and the system becomes (Eckhaus) unstable. Similar behavior could occur in direction solidification [7,8] if the solidification cell is accelerated slowly, rather than pulled at a constant velocity, through a temperature gradient until the (Mullins-Sekerka) instability is encountered and the liquid/solid interface becomes unstable. In each of these scenarios the systems become unstable with respect to fluctuations of certain wavelengths that lie within a band. When the size of the system is comparable to the length scales associated with the wavevectors in the unstable band, the system is described as mesoscopic and the number of accessible states is finite. As illustrated in an extensive review by Cross and Hohenberg [1], instabilities that result in this type of mesoscopic behavior are extremely common, occurring in many diverse fields, such as fluid dynamics, chemical reactions, material science and biology.

The focus of this paper is on state selection in mesoscopic accelerated systems immediately following the onset of the instability. The combination of the driving force that accelerates the system, and the mesoscopic system size that allows for multiple, isolated metastable states, leads to novel and interesting selection rules. It will be shown that the rate at which the system is driven through the instability plays a prominent role in determining the probability that a particular metastable state is selected. As the decay is from states of marginal stability, the selection is also influenced by the noise in the system. The dependences on both acceleration rate and noise strength are considered.

For illustrative purposes consider the situation in which an infinitely long solenoid, carrying a current that increases linearly with time, passes through the center of a narrow superconducting ring of cross-sectional area A and circumference $L \equiv \xi(T)\ell$, where $\xi(T)$ is the temperature dependent correlation length. By Faraday's law of induction, a constant electromotive force (emf) V is induced in the superconductor, thereby accelerating the superconducting electrons. The dynamics of the (dimensionless) superconducting order parameter $\psi(x,t)$, where x is the longitudinal spatial coordinate and t is time, is described by the stochastic time-dependent Ginzburg-Landau equation [4,5]:

$$\partial_t \psi = \partial_x^2 \psi + \psi - \psi|\psi|^2 + i\ell^{-1} x\omega\psi + \eta \qquad (1)$$

where $\omega \equiv \tau_{GL}(2eV/\hbar)$ is a dimensionless measure of the strength of the induced emf. Throughout this paper the regime where $\omega \ll 1$ is considered. In Eq. (1) τ_{GL}, is the Ginzburg-Landau relaxation time, and ψ satisfies the twisted-periodic boundary condition $\psi(\ell + x, t) = \exp(i\omega t)\psi(x,t)$. This equation ignores the influence of the normal current and is valid for a voltage driven system in the limit of low normal state resistivity. The variable η is a Gaussian random variable, with expectation values $\langle \eta(x,t) \rangle = 0$, and $\langle \eta(x,t)\eta^*(x',t') \rangle = 2D\delta(x - x')\delta(t - t')$, where D is determined by the fluctuation-dissipation theorem [9].

For $\omega \ll 1$, the relevant current-carrying states of the superconductor are uniformly twisted plane wave solutions given by $\bar{\psi} = \sqrt{1 - q^2}\exp(iqx)$, where $q = mK + \omega t/\ell$, and $K \equiv 2\pi/\ell$. The dimensionless current density J of these states is given by $J = (\bar{\psi}^*\partial_x\bar{\psi} - \bar{\psi}\partial_x\bar{\psi}^*)/2i = q(1 - q^2)$. Thus the effect of the induced emf (which increases q linearly with time) is to wind the order parameter, or equivalently, to accelerate the superconducting electrons. However, this acceleration cannot continue in-

definitely because J is a nonmonotonic function of q, and hence time, achieving a maximum value of $J_c = 2/\sqrt{27}$ at $q = q_c = 1/\sqrt{3}$. This saturation of the current at the critical current J_c, coincides with the loss of stability of states $\bar{\psi}$ at $q = q_c$. In other words, for $q > q_c$, $d^2F(q)/dq^2 < 0$, where $F(q) \equiv F_{GL}[\bar{\psi}]$ is the Ginzburg-Landau free energy of states $\bar{\psi}$.

To understand the Eckhaus instability for finite size systems it is necessary to perform a linear stability analysis about the state $\bar{\psi}$, as the previous analysis only applies when $\ell = \infty$. Standard linear stability analysis gives one potentially positive eigenvalue that takes the form [10]

$$\lambda_n(q) = -1 + q^2 - k_n^2 + \sqrt{(1-q^2)^2 + 4q^2k_n^2}\ . \qquad (2)$$

The eigenvector associated with this eigenvalue is a linear combination of Fourier modes with wavevector $q \pm k_n$ and amplitude A_n, where $k_n = nK$. The interesting feature of this eigenvalue is that it can become positive when $q > \kappa_1 > q_c$, where $\kappa_m \equiv \frac{1}{\sqrt{3}}[1 + m^2K^2/2]^{1/2}$. Thus, for finite size systems the instability is pushed to wavevectors greater than q_c by an amount that depends on ℓ [10]. In particular, for $\kappa_m > q > \kappa_1$, λ_n is positive for all values of $k_n < mK$ [11]. The dependence of λ on k_n is shown in the inset of Fig. 1 for several values of q.

The growth of a single Fourier mode (with amplitude A_n) of wavevector $q - k_n$, and simultaneous decay of A_0, corresponds to a decrease of the winding number $W = (2\pi)^{-1}\int_0^\ell d\phi(x)/dx$, where ϕ is the phase of ψ, by an amount n. This phenomenon is known as a "phase-slip" as the total phase of the order parameter changes by an integral multiple of 2π. Physically, the supercurrent decreases by a discrete amount when a phase-slip occurs. Phase-slip processes can also occur via thermal activation over an energy barrier and this process has received significant attention over the years [3,4,6,12]. For $D = 10^{-3}$, as long as $\omega \gtrsim 10^{-24}$, the probability of a thermally activated phase slip occurring is exceedingly small [5]. Thus, unless the temperature is very close to the superconducting transition temperature T_c, where D is large, the system will almost always be driven to the Eckhaus instability before a thermally activated phase-slip can occur. Therefore, the transitions that are of concern in this work involve the decay from an unstable state, in contrast to previous work [3–6] where the focus was on the decay from a metastable state.

When $\omega > 0$, the system is driven to the point of instability as the eigenvalues of each Fourier mode eventually become positive. As illustrated in Fig. 1, the $n = 1$ mode becomes unstable first, then the $n = 2$ mode becomes unstable, and so on. This implies that the system first becomes unstable with respect to single phase-slip processes, then double phase-slip processes, etc. If ω is large enough, the $n = 1$ mode might not have time to grow to dominance by the time the $n = 2$ mode becomes unstable. This suggests that for small ω, single phase-slip processes should dominate the dynamics, but as ω is increased there is a crossover to a regime in which double phase-slip processes dominate. As ω is increased further, double phase-slip processes should give way to triple phase-slip processes, and so on.

The generic features displayed in Fig. 1 are common to many systems and come under the general classification scheme of Cross and Hohenberg [1] as type $\mathrm{II_s}$. Thus, the dynamic competition between unstable modes discussed above is a phenomena that has relevance to many systems. The precise determination of which of the modes will initially be selected following the onset of the instability is a complex question that depends on the details of the individual system. For concreteness, the general problem of the dependence of the selected state as a function of the driving force will be addressed for the superconductor. In particular, the relative probability that a given state is selected will be determined by extensive computer simulations. Second, it will be shown that the qualitative features of these results can be understood by an analysis that is based on the properties of the growth rates λ.

To evaluate the probability of the occurrence of a given phase slip as a function of ω, Eq. (1) was numerically integrated in time for a noise strength of $D = 10^{-3}$ and a length corresponding to $n_\ell \equiv \ell q_c/(2\pi) = 5$ [13]. In Fig. 2a, the probability of a type-n phase slip is plotted as a function of ω. As expected, for small ω, single phase-slips dominate. As ω increases further there is a crossover to a regime in which double phase-slips dominate. Further increase of ω results in a subsequent crossover to a regime in which triple phase-slips dominate, and so on.

An example of the dynamics that lead to such results is shown in Fig. 3. In this figure, the winding number and current are plotted as a function of time for $\omega = 5 \times 10^{-4}$. This value of ω is in the crossover region between the single and double phase-slip dominated regimes. Clearly evident in this figure are the single and double phase slips in which W changes by one or two, respectively. Also seen in Fig. 3b are the discrete jumps of the supercurrent.

As described earlier, the essential features shown in Fig. 2a can be understood using the properties of the growth rates λ_n. This idea can be made more concrete in the following way. Ignoring the nonlinear interactions between the different modes, the expectation of $|A_n|^2$ is given by [5]

$$\langle |A_n(t)|^2 \rangle = \frac{2D}{\ell} e^{2\sigma_n(t)} \int_0^t dt'\, e^{-2\sigma_n(t')}, \qquad (3)$$

where $\sigma_n(t) \equiv \int_0^t dt'\, \lambda_n(q(t'))$, and angular brackets denote a noise average. After the onset of the instability the system evolves towards the fixed points $\bar{\psi}_n = \bar{A}_n \exp(i(q - nK)x)$, where $\bar{A}_n = \sqrt{1-(q-nK)^2}$. Eq. (3) describes the initial evolution of the system after the Eckhaus boundary has been reached. In this non-interacting picture each amplitude (measured in units of $\bar{A}_n$) can be thought of as an orthogonal coordinate in an n_ℓ-dimensional space. Thus, the natural measure of distance from the origin ($A_n = 0$) in this space is

$\sum_{n=1}^{n_\ell}\langle|A_n(t)|^2\rangle/\bar{A}_n^2$. After onset of the Eckhaus instability, this sum increases rapidly and reaches unity at a time t^*. Assuming that at $t=t^*$ a phase-slip has occured with probability one it is natural to interpret $\langle|A_n(t^*)|^2\rangle/\bar{A}_n^2$ as the relative probability of the occurrence of a type-n phase-slip. The probabilities calculated using this procedure are shown in Fig. 2b.

It is clear from Fig. 2 that the preceding analysis provides a qualitatively accurate description of the state-selection probabilities, and their dependence on the driving force ω. Most notably, the values of ω at the peak positions agree very well with the numerical results. Nevertheless it is important to point out that the preceding analysis is only a plausible argument and is not systematic. A quantitative description must include the subtle non-linear interactions that are an important element in determining state selection. Even at the present level of ignorance, however, the analysis presented here provides a qualitatively useful description of the state-selection probabilities, and their dependence on the driving force.

The growth rates λ are an extremely important factor in determining the state selection probabilities. The preceding analysis accounts for these growth rates and therefore provides a qualitatively accurate description. The analysis also provides predictions for the dependence of P_n on the noise strength D, which may be more convenient to vary in some experiments. Plotted in Fig. 4a are the probabilities of a type-n phase slip as a function of D, for a fixed value of ω, obtained from a numerical simulation of Eq. (1). In Fig. 4b, the corresponding P_n's obtained from the growth-rate analysis are plotted for comparison. Once again, it is seen that the simple analysis provides an accurate qualitative picture. For the smallest values of D considered triple phase-slip processes dominate. This is because the time required for a given mode to grow to saturation diverges logarithmically as $D\to 0$. Consequently, if D is very small, the mode amplitudes A_1 and A_2, for example, may still be very small by the time the growth rate of A_3 is significantly larger than the growth rates for A_1 or A_2.

One of the most interesting aspects of the phenomena exposed here is that the selection rules depend on both the intrinsic properties of the system and the external parameters. To understand this connection more deeply, it is instructive to consider the characteristic growth times for individual modes. Typically, the characteristic time associated with the initial growth of an unstable mode is taken to be the inverse of the growth rate. However, for accelerated systems the growth rate λ starts out negative and passes through zero. Thus, $|\lambda^{-1}|$ is not a relevant quantity as it diverges at the instability. To determine the characteristic time, consider Eq. (3) for $\langle|A_n(t)|^2\rangle$. The quantity $\sigma(t)$ achieves a local minimum at $t=t_n\equiv\ell\kappa_n/\omega$ so that a second order expansion about t_n yields $\sigma_n(t)\approx\sigma_n(t_n)+\frac{1}{2}\frac{\lambda_n'\omega}{\ell}(t-t_n)^2$, where $\lambda_n'\equiv\partial\lambda_n/\partial q|_{q=\kappa_n}$. Inserting this expansion into Eq. (3) and assuming that $\omega\ll 1$ gives

$$\langle|A_n(t)|^2\rangle = 2D\tau_n\ell^{-1}\exp\left(z_n^2(t)\right)\left[\operatorname{erf}\left(z_n(t)\right)+1\right], \quad (4)$$

where $z_n(t)=(t-t_n)/\tau_n$ and $\tau_n=\sqrt{\ell/\lambda_n'\omega}$. The quantity τ_n is the characteristic time for the growth of mode-n, and is interesting because it depends on the geometric mean of λ_n' and ω. Thus, the time scale τ_n embodies in a natural way the importance of the combination of the intrinsic dynamics (λ_n') and the external driving force (ω).

In summary, the problem of state selection in accelerated systems has been shown to contain unique and rich phenomenology. Although the focus of this work has been on the dynamics of quasi-one-dimensional superconducting rings, the essential features of this particular system are generic and should be observable in a diverse array of experimentally realizable situations. Despite the success of the linear analysis, it is clear that new methods must be developed to explore this complex and important area of research in nonequilibrium statistical mechanics. Recent work [14] on state selection in non-accelerated marginally stable systems suggests a possible systematic framework that could be extended to address the phenomena considered here.

We thank Paul Goldbart for suggesting the problem of nonequilibrium superconductivity, and Paul Goldbart and Alan McKane for useful discussions. This work was supported in part by the MRSEC program of the NSF (DMR-9400379) (MBT), Research Corporation Grant #CC4181 (KRE), and grant NSF-DMR-8920538 administered through the U. of Illinois Materials Research Laboratory (MBT, KRE).

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of wavevector $q + k$, which grow initially according to linear analysis, will be damped by the nonlinear terms. Therefore, these modes act as catalysts in that they play an active role in the growth of the $q - k$ modes, but they themselves cannot compete for occupation as they are unstable.

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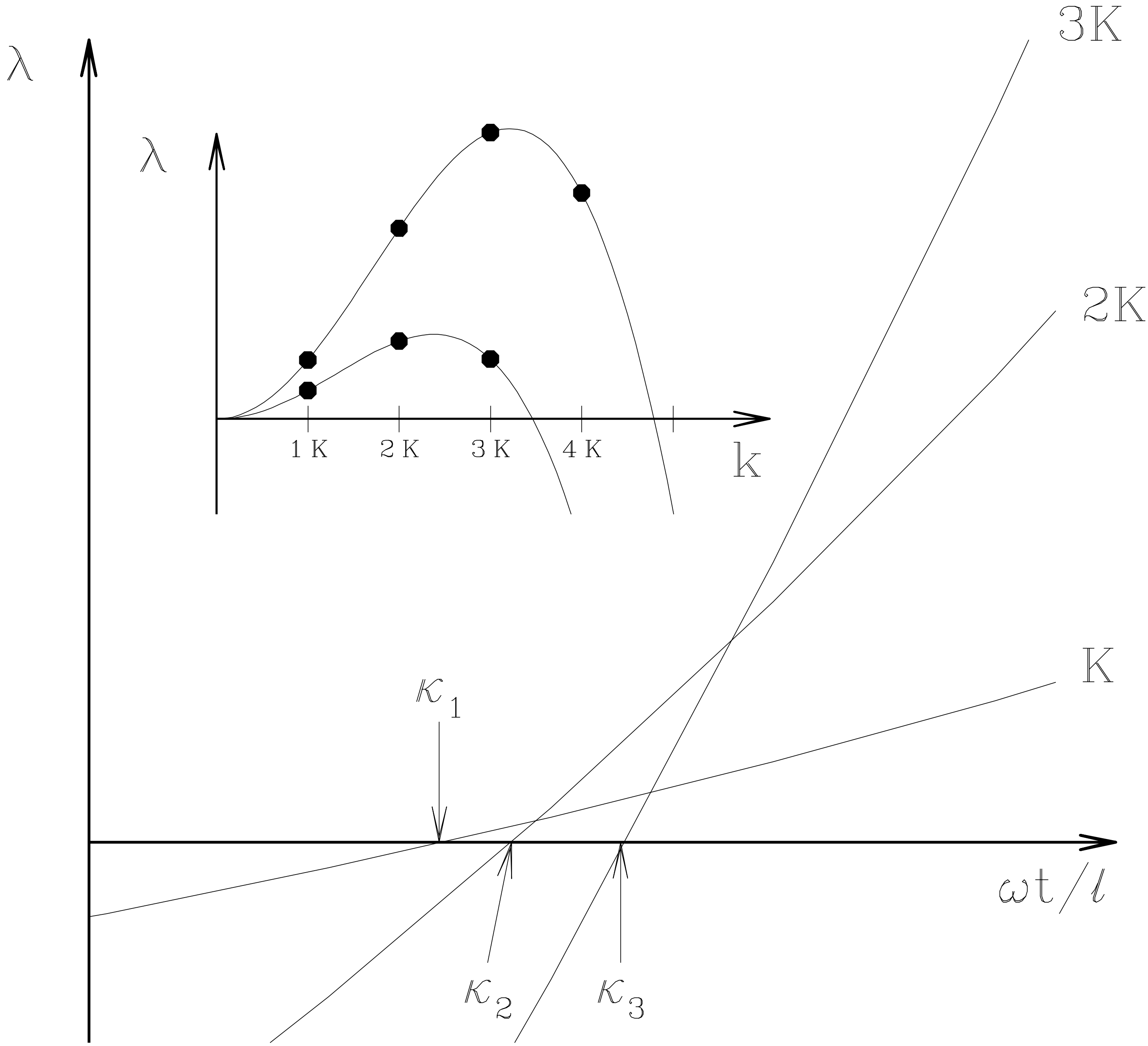
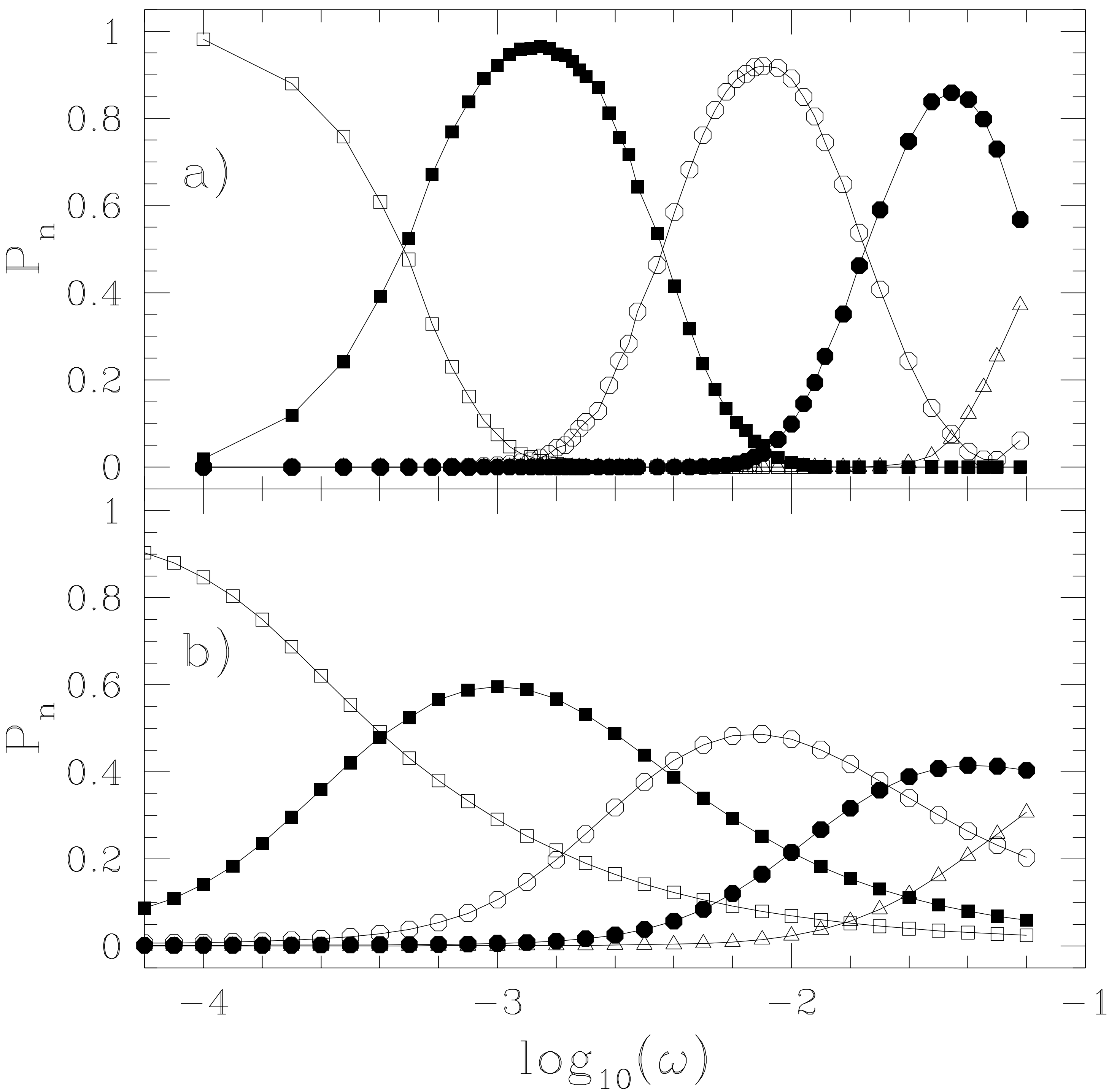
FIG. 1. λ as a function of $q = \omega t/\ell$ for $k = K, 2K, 3K$. Inset: λ as a function of k for two values of $q > \kappa_1$ such that the upper curve corresponds to the larger value of q.

FIG. 2. State selection probabilities as a function of the driving force ω. Open squares, solid squares, open circles, solid circles and open triangles correspond to the probabilities P_1, P_2, P_3, P_4 and P_5, respectively. Results of the numerical integration of Eq. (1) and those of the linear analysis described in the text are shown in Figs. (a) and (b), respectively.

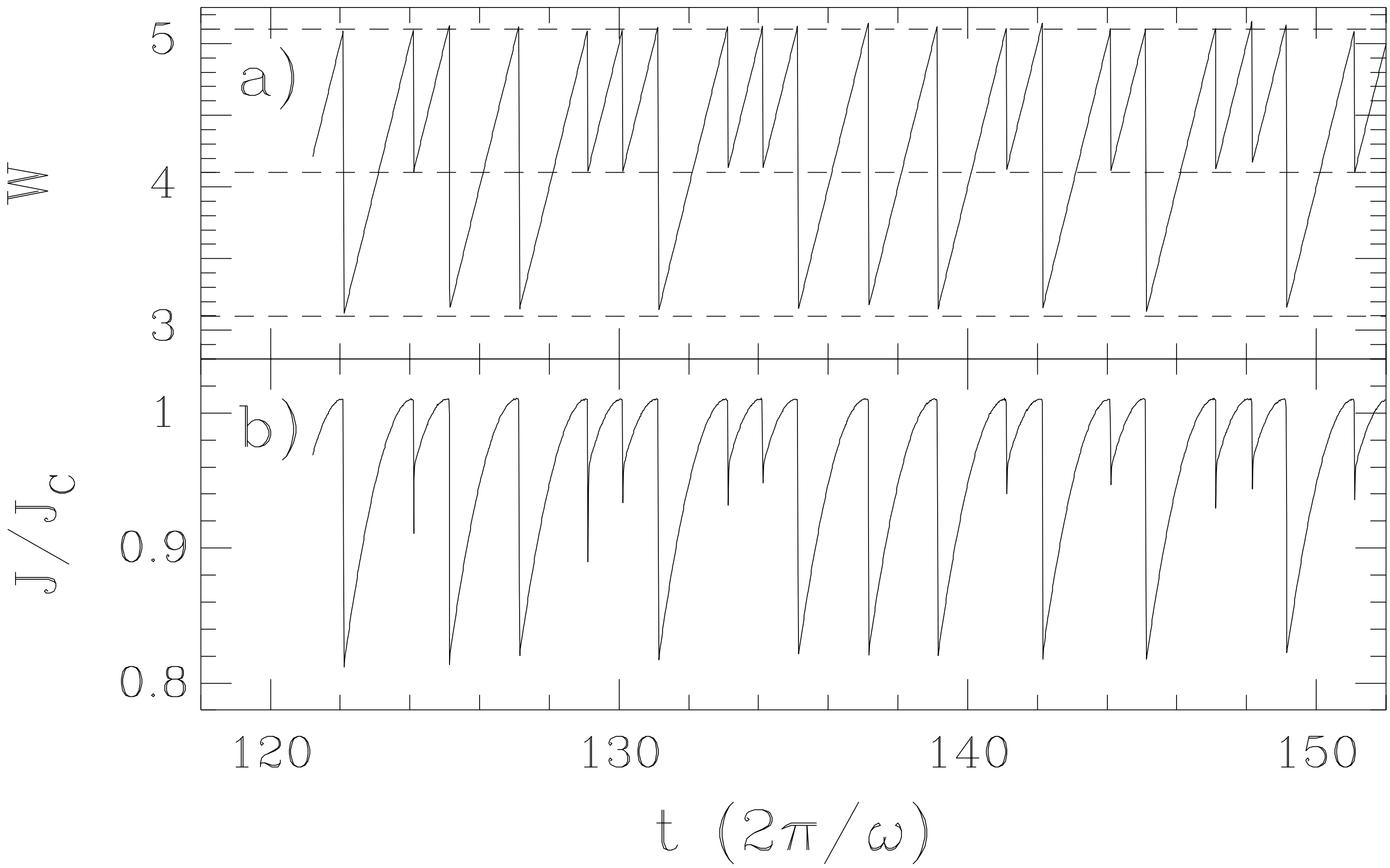
FIG. 3. Dynamics of winding number (a) and supercurrent (b), for $\omega = 5 \times 10^{-4}$ and $D = 10^{-3}$.

FIG. 4. State selection probabilities as a function of the noise strength D. The symbols in this figure are identical to those in Fig. 2. Results of the numerical integration of Eq. (1) and those of the linear analysis described in the text are shown in Figs. (a) and (b), respectively.

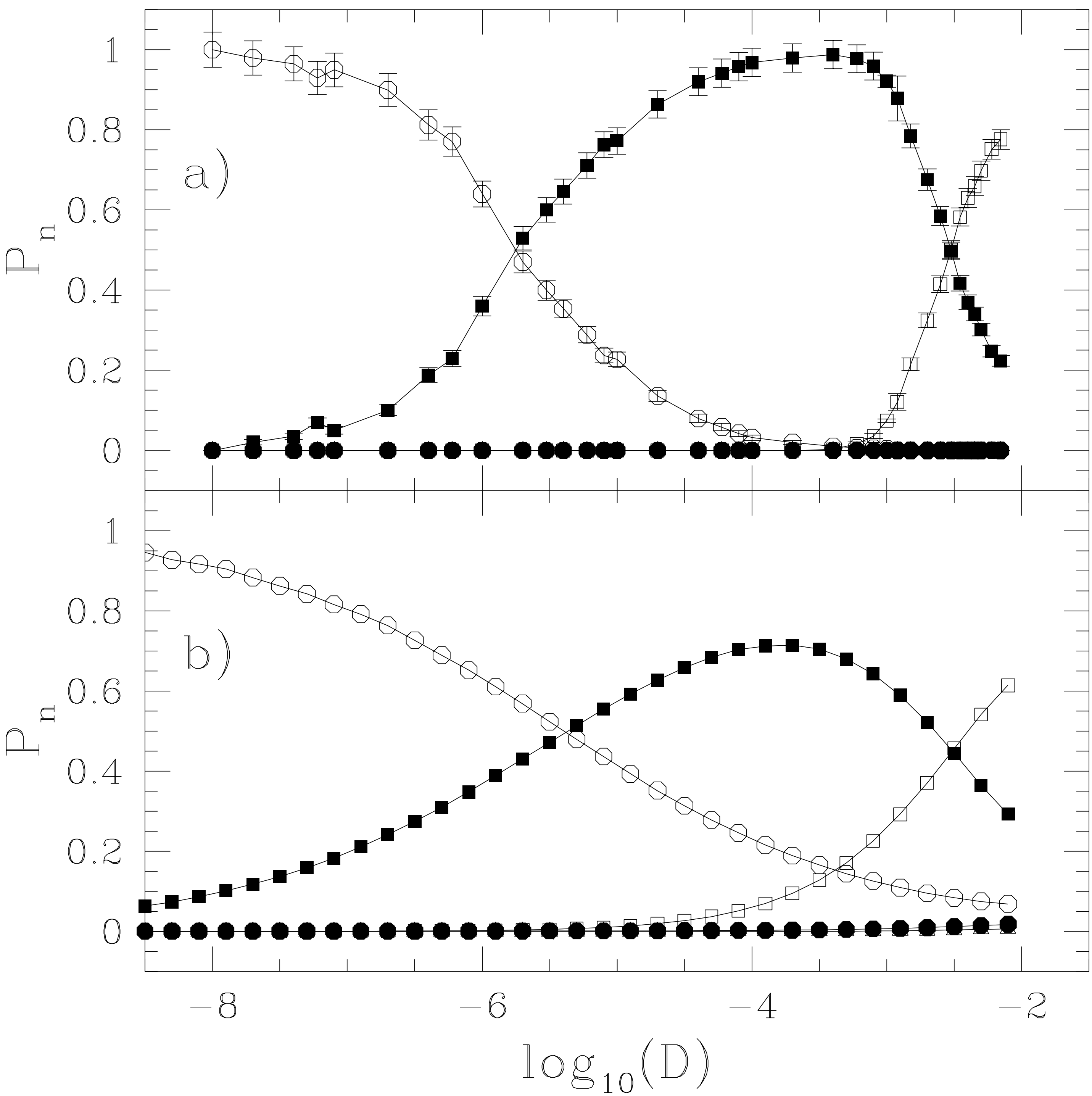

λ
λ
1 K
2 K
3 K
4 K
k
3K
2K
K
κ1
κ2
κ3
ωt/ℓ



a)
b)
P_n
P_n
1
0.8
0.6
0.4
0.2
0
−4
−3
−2
−1
$\log_{10}(\omega)$



W
5
4
3
a)
J / J_c
1
0.9
0.8
b)
120
130
140
150
t (2π/ω)



a)
b)
P_n
P_n
0
0.2
0.4
0.6
0.8
1
−8
−6
−4
−2
$\log_{10}(D)$